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Markov Operators on Hermitian 2×2 Matrix Spaces with p-order

M.A. Berdikulov
Institute of Mathematics and Information Technologies,
Uzbekistan Academy of Sciences,
29, F. Hodjaev str., Tashkent 100125, Uzbekistan
E-mail: musulmonqul@mail.ru

ABSTRACT

In this work the Markov operators on the space of 2×2 matrices with p-order are studied. In this case the general form of the Markov's operators is found. Sufficient conditions for an operator to be Markov in this spaces given. The regularity, accuracy and periodicity of the Markov's operator is investigated.

Keywords: Markov operator, Hermitian matrix, regularity, accuracy, periodicity, p -order.

INTRODUCTION

A Markov chain on matrix spaces is an analog of a classical Markov chain with finite number of states [1]. It is known [2], that the statistical model of the quantum mechanics can be constructed on the base of a von Neumann algebra which is the algebra of linear bounded operators in a Hilbert space.

It appears, problems of probability theory can be considered on non-algebraic structures, for example, on ordered normed spaces [3].

It is known, that a Hermitian matrix is positively defined if it is the square of another Hermitian matrix or *eigenvalues* of the matrix positive numbers.

In [4], it is given a new definition of positively definition for a 2×2 Hermitian matrix named p- positively defined, for $p>1$, which is not coordinated with an algebraic structure at $p\neq2$. That is there exists a Hermitian matrix, the square of which is not p-positively defined. A matrix space with the p-order isn't a normed algebra with respect to p-order norm. The p-order norm coincides with the operator norm in the case of

$p=2$. The space of 2×2 Hermitian matrices is an order-unit space (see definition in [5]) with the p-order.

Let $M_2(C)$ be the algebra of complex 2×2 matrices. We designate the set of Hermitian matrices as $M_2(C)_{sa}$. An element T of $M_2(C)_{sa}$ has the form $T=\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$.

Definition 1. Let $p>1$. The following numbers

$$\lambda_{1,2}=\frac{1}{2}\left(a+d\pm\left(|2b|^p+|2c|^p+|a-d|^p\right)^{\frac{1}{p}}\right)$$

are said to be p-*eigenvalues* of the matrix T.

The numbers $\lambda_{1,2}$ are studied in details in [4] and it was shown that if $p=2$, these numbers coincide with usual eigenvalues of the matrix T.

Definition 2. A matrix $T\in M_2(C)_{sa}$ is called p-*positively defined* if $\lambda_{1,2}\geq 0$. We'll write in this case $T\geq_p \theta$ where θ is the zero-matrix.

If T is a p-positively defined matrix then it is obvious that $a\geq 0$, $d\geq 0$.

We say that $T\geq_p S$ for $T,S\in M_2(C)_{sa}$ if $T-S\geq_p \theta$. We designate the set of p-positively defined matrices as M_p^+, spaces $M_2(C)_{sa}$ with p-order as $\mathrm{A}=M_p^2(C)_{sa}$.

Analogously to [4], one can show that M_p^+ is the generating cone in the space $M_2(C)_{sa}$. The unit matrix E will be p-order unit in $M_p^2(C)_{sa}$ and the following norm

$$\|T\|_p=\frac{1}{2}\left(|a+d|+\left(|2b|^p+|2c|^p+|a-d|^p\right)^{\frac{1}{p}}\right),$$

coincides with the p-order norm in $M_2(C)_{sa}$.

In fact, for a nonnegative number λ the inequalities $-\lambda E \leq_p T \leq_p \lambda E$ means

$$-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \leq_p \begin{pmatrix}a & b+ic\\ b-ic & d\end{pmatrix} \leq_p \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Consider the inequality: $\begin{pmatrix}a-\lambda & b+ic\\ b-ic & d-\lambda\end{pmatrix} \leq_p \theta$.

By definition 1, $a+d-2\lambda \pm \left(|2b|^p + |2c|^p + |a-d|^p\right)^{\frac{1}{p}} \leq 0$.

Hence, $\lambda \geq \frac{1}{2}\left(a+d \pm \left(|2b|^p + |2c|^p + |a-d|^p\right)^{\frac{1}{p}}\right)$.

Analogous, the inequality $\begin{pmatrix}a+\lambda & b+ic\\ b-ic & d+\lambda\end{pmatrix} \geq_p \theta$ follows $a+d+2\lambda \pm \left(|2b|^p + |2c|^p + |a-d|^p\right)^{\frac{1}{p}} \geq 0$.

Hence, $\lambda \geq \frac{1}{2}\left(-a-d \pm \left(|2b|^p + |2c|^p + |a-d|^p\right)^{\frac{1}{p}}\right)$. So, we has $\lambda \geq \frac{1}{2}\left(|a+d| + \left(|2b|^p + |2c|^p + |a-d|^p\right)^{\frac{1}{p}}\right)$. The infimum of these numbers is said to be the order norm of the matrix T [5].

Remark 1. It is known the L_p – norm of a matrix is defined with the help of the trace of the matrix. The introduced norm and the L_p – norm are not connected among themselves.

Remark 2. The space $M_2(C)_{sa}$ is not a normed algebra with respect to this norm since the axiom of normed algebra: $\|TS\| \leq \|T\| \cdot \|S\|$ is not valid.

For example, consider the matrix $T = \begin{pmatrix}1 & 4\\ 4 & 3\end{pmatrix}$. Let take $p = 3$. Then $\|T\|_3 = 2 + \sqrt[3]{65}$, $\|T^2\|_3 = 21 + 4\sqrt[3]{65}$ and $\|T^2\|_3 > \|T\|_3^2$, but this contradicts the axiom $\|TS\|_3 \leq \|T\|_3 \cdot \|S\|_3$ of normed algebra.

The matrix $T=\begin{pmatrix} 1 & \sqrt[3]{7} \\ \sqrt[3]{7} & 3 \end{pmatrix}$ is 3-positively defined, but it is not positively defined in the usual sense. Let $A=M_p^2(C)_{sa}$.

Definition 3. A linear operator $P:A\to A$ is said to be *Markov operator* if it possesses the following properties:

(i) $P(E)=E$, here E is the unit matrix: $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii) $P(T)\geq_p \theta$ if $T\geq_p \theta$.

MAIN RESULTS

It is not difficult to see that a composition of Markov operators is a Markov operator.

Consider examples.

1. $P:\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}\to\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

2. $P:\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}\to\begin{pmatrix} \frac{a+d}{2} & b+ic \\ b-ic & \frac{a+d}{2} \end{pmatrix}$.

3. Let $t,\ t',\ t'',\ s,\ s',\ s''$ be such real numbers that $|t|^p+|t'|^p+|t''|^p=1$, $|s|^q+|s'|^q+|s''|^q=1$, where $p\geq 1$, $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$ satisfying the condition $ts+t's'+t''s''=1$.

Set $P:\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$

$$\to\frac{1}{2}\begin{pmatrix} a+d+((a-d)s+(bs'+cs''))t & ((a-d)s+2(bs'+cs''))(t'+it'') \\ ((a-d)s+2(bs'+cs''))(t'-it'') & a+d-((a-d)s+2(bs'+cs'')t) \end{pmatrix}.$$

The Gelder inequality follows the p - positively defined of this operator.

Note that a Markov operator P can be represented as a 4×4-matrix.

Theorem 1. Matrix of the Markov operator P has the form:

$$P=\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{13} \\ p_{21} & p_{22} & p_{23} & p_{23} \\ p_{31} & -p_{31} & p_{33} & p_{34} \\ p_{31} & -p_{31} & p_{34} & p_{33} \end{pmatrix} \tag{1}$$

with conditions $p_{ij}\geq 0,\ i,j=1,2,\ p_{11}+p_{12}=1,\ p_{21}+p_{22}=1$ and $|2p_{31}|^p\leq \min\{|p_{11}+p_{21}|^p-|p_{11}-p_{21}|^p, |p_{12}+p_{22}|^p-|p_{12}-p_{22}|^p\}.$

Proof. We consider in the $M_2(C)$ a standard basis $e_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $e_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, $e_3=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $e_4=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Let $(p_{ij})_{i,j=1}^4$ be the matrix of the operator $P.$ We are confined to the case when coefficients p_{ij} are real.

Hence, $e_1\geq_p \theta.$ By the property (ii) of a Markov operator, $Pe_1\geq_p \theta.$ On the other hand, $Pe_1=p_{11}e_1+p_{21}e_2+p_{31}e_3+p_{41}e_4.$ So,

$$Pe_1=\begin{pmatrix} p_{11} & p_{31} \\ p_{41} & p_{21}\end{pmatrix}$$

is p-positively defined, i.e.

$$p_{31}=p_{41},\quad p_{11}\geq 0,\quad p_{21}\geq 0,\quad p_{11}+p_{21}\geq \left(|2p_{31}|^p+|p_{11}-p_{21}|^p\right)^{\frac{1}{p}}. \tag{2}$$

Analogously we obtain with the help of e_2

$$p_{32}=p_{42},\quad p_{12}\geq 0,\quad p_{22}\geq 0,\quad p_{12}+p_{22}\geq \left(|2p_{32}|^p+|p_{12}-p_{22}|^p\right)^{\frac{1}{p}}. \tag{3}$$

Further, the unit matrix E can be written in the form: $E=e_1+e_2.$ Hence,

$$P(E)=Pe_1+Pe_2=(p_{11}+p_{12})e_1+(p_{21}+p_{22})e_2+(p_{31}+p_{32})e_3+(p_{41}+p_{42})e_4=E.$$

We have

$$p_{11} + p_{12} = 1, p_{21} + p_{22} = 1, p_{31} + p_{32} = 0, p_{41} + p_{42} = 0, \text{i.e. } p_{32} = -p_{31}. \quad (4)$$

Since P transfers a Hermitian matrix to a Hermitian one, the condition $(Pe_3)^* = Pe_4$ should be realized. It implies that $p_{13} = p_{14}$, $p_{43} = p_{34}$, $p_{33} = p_{44}$, $p_{23} = p_{24}$. ∇

Remark 3. It is not difficult to see that for p=2 formulas (2), (3) and (4) coincide with the correspondent formulas (2), (3) and (6) of the work [1] for 2×2 matrices.

Limit behavior investigation of the iterations P^n for the given operator P at $n \to \infty$ is one of the basic problems in Markov operators theory. The following result takes place.

Theorem 2. Let P be a Markov operator in A. Then

- (i) $\|P(T)\|_p \leq \|T\|_p$ for any $T \in A$;
- (ii) for any $T \in A$, the sequence $P^n(T)$, $n = 1, 2, \ldots$ is bounded by the norm;
- (iii) matrix elements p_{ij}^n of the operator P^n, $n = 1, 2, \ldots$ are bounded in the all;
- (iv) if $\lambda \in C$ is an eigenvalue of the operator P then $|\lambda| \leq 1$.

Proof. (i) Let $T \in A$. Since E is the p-order unit, then

$$-\|T\|_p E \leq_p T \leq_p \|T\|_p E.$$

If we applies the operator P to these inequalities, then we obtain

$$-\|T\|_p E \leq_p P(T) \leq_p \|T\|_p E.$$

It is means that $\|P(T)\|_p \leq \|T\|_p$.

(ii) Since the operator P^n is a Markov operator, then (i) implies that $\|P^n(T)\|_p \leq \|T\|_p$ for all $n = 1, 2, \ldots$.

(iii) follows (iii).

(iv) Let $T \in A$, $T \neq \theta$, $\lambda \in C$, $P(T) = \lambda T$. Then $P^n(T) = \lambda^n T$, $n = 1, 2, \ldots$. Hence, $\left\| P^n(T) \right\|_p = |\lambda|^n \left\| T \right\|_p$. As $\left\| T \right\|_p \neq 0$ and the sequence $\left\| P^n(T) \right\|_p$, $n = 1, 2, \ldots$, is bounded, $|\lambda| \leq 1$.

Definition 4. We say that a Markov operator P given on A is *regular* if there exists a state μ on A such

$$\lim_{n \to \infty} P^n(T) = \mu(T)E$$

for any $T \in A$. Here, the limit is taken by the p-order norm.

Theorem 3. Let P be a Markov operator in A. The operator P is regular if and only if eigenvalues of this operator satisfy the condition $\lambda_1 = 1$, $|\lambda_i| < 1$, $i = 2, 3, 4$.

As matrix positively definition doesn't play any role by proof of this theorem, one can prove it similar to theorem 1 in [1].

Definition 5. A Markov operator P is said to be *accurate* if the limits

$$\lim_{n \to \infty} P^n(T)$$

exists for any $T \in A$.

Theorem 4. Let P be a Markov operator in A. The operator P is accurate if and only if eigenvalues of this operator satisfy the conditions $\lambda_1 = 1$, $|\lambda_i| \leq 1$, $i = 2, 3, 4$.

Proof of this theorem is analogous to proof of theorem 2 in [1].

Definition 6. A Markov operator P is called *periodical* with the period $d > 1$ if

a) P^d is accurate;
b) the number d is the least among the integers for which the condition a) holds.

Corollary 1. A Markov operator P is periodical with the period d if and only if

a') all the eigenvalues of the operator P, modulus of which are 1, are roots of the power d from unit;

b') the number d is the least among the integers for which the condition a') holds.

Markov operators given in examples 1 and 2 are periodical with the period 2 and the matrices corresponding to them operators have the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{and} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We give sufficient conditions at which a matrix in the form (1) determines a Markov operator in $A = M_p^2(C)_{sa}$.

Matrices

$$U = \frac{1}{2}\begin{pmatrix} 1+t & t' \\ t' & 1-t \end{pmatrix}, \tag{5}$$

where $t,\ t'$ are real numbers such, that $|t|^p + |t'|^p = 1$, are basic matrices in $M_p^2(C)_{sa}$ [4].

It is sufficient for an operator P to be Markov operator that for any matrix U in the form (5), the matrix

$$P(U)$$

$$= \frac{1}{2}\begin{pmatrix} (1+t)p_{11} + (1-t)p_{12} + 2p_{13}t' & (1+t)p_{31} - (1-t)p_{31} + (p_{33} + p_{34})t' \\ (1+t)p_{31} - (1-t)p_{31} + (p_{33} + p_{34})t' & (1+t)p_{21} + (1-t)p_{22} + 2p_{23}t' \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + (p_{11} - p_{12})t + 2p_{13}t' & 2p_{31}t + (p_{33} + p_{34})t' \\ 2p_{31}t + (p_{33} + p_{34})t' & 1 + (p_{21} - p_{22})t + 2p_{23}t' \end{pmatrix},$$

is p -positively determined. As $P(U) \geq_p \theta,$ we have

$$1+(p_{11}-p_{12})t+2p_{13}t' \geq 0, \tag{6}$$

$$1+(p_{21}-p_{22})t+2p_{23}t' \geq 0, \tag{7}$$

$$(2+(p_{11}-p_{12}+p_{21}-p_{22})t+2(p_{13}+p_{23})t')^p \geq$$

$$|4p_{31}t+2(p_{33}+p_{34})t'|^p+|(p_{11}-p_{12}-p_{21}+p_{22})t+2(p_{13}-p_{23})t'|^p .$$

Since t and t' are arbitrary, inequalities (6), (7) are equivalent to the following inequalities

$$|(p_{11}-p_{12})t+2p_{13}t'|\leq 1, \quad |(p_{21}-p_{22})t+2p_{23}t'|\leq 1, \tag{8}$$

respectively. They respectively follow by the Gelder inequality from the following inequalities:

$$|p_{11}-p_{12}|^q+|2p_{13}|^q \leq 1, \quad |p_{21}-p_{22}|^q+|2p_{23}|^q \leq 1. \tag{9}$$

Here q is the number determined from the equality $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Analogously, the inequality

$$\begin{aligned}&(2+(p_{11}-p_{12}+p_{21}-p_{22})t+2(p_{13}+p_{23})t')^p \geq \\ &\left(|4p_{31}|^q+|2p_{33}+p_{34}|^q\right)^p+\left(|p_{11}-p_{12}-p_{21}+p_{22}|^q+|2(p_{13}-p_{23}|^q\right)^p\end{aligned} \tag{10}$$

follows inequality (8).

Arbitrariness of t and t', and equalities $p_{11}+p_{12}=1, \;\; p_{21}+p_{22}=1$ follow

$$\begin{aligned}&(1+p_{13}+p_{23})^p \geq (|2p_{31}|^q+|(p_{33}+p_{34}|^q)^p \\ &+(|1-p_{12}-p_{21}|^q+|p_{13}-p_{23}|^q)^p,\end{aligned} \tag{11}$$

$$\begin{aligned}&(2-p_{12}-p_{22})^p \geq (|2p_{31}|^q+|(p_{33}+p_{34}|^q)^p \\ &+(|1-p_{12}-p_{21}|^q+|p_{13}-p_{23}|^q)^p.\end{aligned} \tag{12}$$

Thus, we have proved the following

Theorem 5. Let P be a linear operator in A determined by the matrix (1). If elements of P satisfy the relations (9), (11), (12), then P is Markov operator. Moreover, the following result takes place.

Theorem 6. If elements of P satisfy the strict version of the inequalities (9), (11) and (13) then P is a regular Markov operator.

Let P be a Markov operator determined by the matrix (1). Consider the characteristic equation

$$det(p_{ij} - \lambda\delta_{ij}) = 0.$$

Add to the first column of the determinant $det(p_{ij} - \lambda\delta_{ij})$ the second one. Then

$$\begin{vmatrix} 1-\lambda & p_{12} & p_{13} & p_{13} \\ 1-\lambda & p_{22}-\lambda & p_{23} & p_{23} \\ 0 & -p_{31} & p_{33}-\lambda & p_{34} \\ 0 & -p_{31} & p_{34} & p_{33}-\lambda \end{vmatrix} = 0,$$

hence one can obtain after non-complicated transformations

$$(1-\lambda)(p_{33} - p_{34} - \lambda))[(p_{22} - p_{12} - \lambda)(p_{33} + p_{34} - \lambda) + 2p_{31}(p_{23} - p_{13})] = 0.$$

Suppose that even if one form the two conditions

$$p_{23} = p_{13}, \qquad p_{31} = 0$$

holds. In this case, eigenvalues of the operator P have the following form:

$$\lambda_1 = 1, \quad \lambda_2 = p_{22} - p_{12}, \quad \lambda_{3,4} = p_{33} \pm p_{34}.$$

By theorem 2, $|\lambda_i| \le 1$, $i = 1,2,3,4$. Therefore only the following cases are possible:

1. $|p_{22} - p_{12}| < 1$, $|p_{33} \pm p_{34}| < 1$. By theorem 3, the operator P is regular.
2. $p_{22} - p_{12} \ne -1$, $p_{33} \pm p_{34} \ne -1$. By theorem 4, the operator P is accurate.
3. Even if one from the two numbers $p_{22} - p_{12}$, $p_{33} \pm p_{34}$ equals 1. In this case by corollary 1, the operator P is periodical of the period 2.

Remark 4. The last results are based only on the view of elements a Markov operator. They coincide with results of the work [1] and are given here for compactness of results.

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